Final Project

The final project draws together concepts from across the quarter: graph algorithms, divide-and-conquer algorithms, randomized algorithms, greedy algorithms, dynamic programming, and intractability. The problems here are designed to combine these topics in new ways so that you can appreciate how versatile a skillset you've acquired this quarter.

Choose and answer one of the six algorithmic problems given here. Each problem may combine two or three different techniques from the course, so by answering two of the three problems you will have demonstrated a mastery of four of the six topics we have covered. You will not improve your overall score by submitting answers to multiple problems – if you do, we will only grade the first two – but you are welcome to answer all three problems and submit your answers to the two you are most comfortable with.

The project must be completed in groups of at most four students. While we permit collaboration on the problem sets, group members **must not** collaborate with anyone else on this project. You can ask the course staff clarifying questions about the problems if you are unsure what they are asking, but we will not provide hints, check your work, etc.

You must not consult any outside resources when completing this project. You may only refer to materials on the course website, the book *Algorithm Design* by Kleinberg and Tardos, notes that you yourself have taken over the course of the class, lecture videos, and your own graded problem sets. For example, you **must not** use a search engine to look up anything related to any of the problems in this project, nor should you look at any other student's notes.

You must submit your answers to the *elearn* portal containing the following artifacts:

1. The implementation of all required algorithms (any programming language)
2. Documentation (description of proposed implementation, time and space complexity, answer to questions, teamwork, etc.)

3. A recorded presentation (15-20 minutes demonstrating the execution of the developed programs as well as describing your solutions and algorithmic decisions).

This final project is worth 20-25% of your grade in this course. The final project's due date is one week after the final exam was held. No late submissions are accepted.

It has been a pleasure teaching Algorithm Analysis and Design this quarter. Best of luck with the final project!

Final Project Topics

Topic 1: Multicolored Spanning Trees

Suppose that you have a connected, undirected graph $G = (V, E)$ where each edge is colored either red or blue. Given a number k, you are interested in determining whether there is some spanning tree of G that contains exactly k blue edges.

i. Design a polynomial-time algorithm that finds a spanning tree of G containing the *minimum* possible number of blue edges. Then:
   - Describe your algorithm.
   - Prove that your algorithm finds a spanning tree of G containing the minimum possible number of blue edges.
   - Prove that your algorithm runs in polynomial time.

ii. Design an algorithm that finds a spanning tree of G containing the *maximum* possible number of blue edges. Then:
   - Describe your algorithm.
   - Prove that your algorithm finds a spanning tree of G containing the maximum possible number of blue edges.

- Prove that your algorithm runs in polynomial time.

iii. Suppose T_1 and T_2 are spanning trees of G where T_1 contains k_1 blue edges and T_2 contains $k_2 > k_1$ blue edges. Prove there must be some spanning tree T of G containing exactly $k_1 + 1$ blue edges.

iv. Design an algorithm that, given a number k, determines whether there is a spanning tree of G that contains exactly k blue edges. Note that you don't need to *find* such a spanning tree; you just need to determine whether one exists. Your algorithm should run in time polynomial in n and m (the number of nodes and edges in G), but not in k. Then:

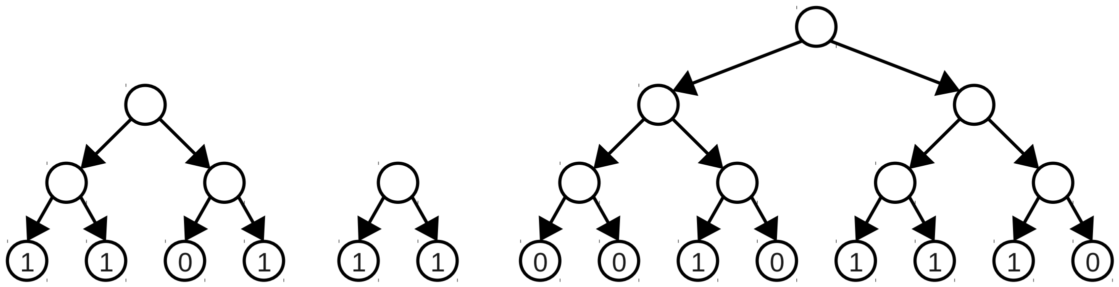
- Describe your algorithm.
- Briefly justify why your algorithm determines whether there is a spanning tree of G containing exactly k blue edges. You don't need to write a formal proof here but should give a one-paragraph justification as to why your algorithm works.
- Briefly justify why your algorithm runs in time polynomial in n and m.

Topic 2: Evaluating NAND Trees

A *NAND tree* is a complete binary tree with the following properties:

- Each leaf node is labeled either 0 or 1.
- All internal nodes are *NAND gates*. A NAND gate is a logic gate that takes in two inputs and evaluates to 0 if both its inputs are 1 and to 1 if either input is 0.

We can *evaluate* a NAND tree by computing the value of the top-level NAND gate in the tree, which will evaluate either to 0 or to 1. (If the tree is a single leaf, the tree evaluates to the value of that leaf.) For example, the left and right trees below evaluate to 1; the middle tree evaluates to 0:



Here is a simple recursive algorithm for evaluating a NAND tree:

- If the tree is a single leaf node, return the value of that node.
- Otherwise, recursively evaluate the left and right subtrees, then apply the NAND operator to both of those values.

This algorithm takes Θ(n) time to evaluate a NAND tree with n-leaf nodes. We can improve this algorithm using *short-circuiting*. If one subtree of node v evaluates to 0, then v must evaluate to 1 because 0 NAND 0 = 1 and 0 NAND 1 = 1. Therefore, we don't need to evaluate v's other subtree. This gives the following algorithm, which we'll call the *left-first algorithm*:

- If the tree is a single leaf node, return the value of that node.
- Otherwise:
- Recursively evaluate the left subtree.
- If it evaluates to 0, return 1.
- Otherwise, recursively evaluate the right subtree.
- If it evaluates to 0, return 1; otherwise, return 0.

In many cases, the left-first algorithm runs faster than the $\Theta(n)$-time naïve algorithm. However, it is possible to construct NAND trees for which the left-first algorithm runs in time $\Theta(n)$.

i. Design an algorithm that creates a NAND tree T with $n = 2^k$ leaf nodes such that the left-first algorithm never short-circuits when evaluating T. Your algorithm should run in time polynomial in n. Then:
   - Describe your algorithm.
   - Prove that your algorithm produces a tree T with n leaves such that the left-first algorithm never short-circuits when evaluating T.
   - Prove your algorithm runs in time polynomial in n.

Since the left-first algorithm never short-circuits on inputs produced by your algorithm, the left-first algorithm has a worst-case runtime of $\Theta(n)$.

More generally, *any* deterministic algorithm for evaluating a NAND tree will have at least one input that causes it to run in $\Theta(n)$ time, but you don't need to prove this.

Despite the $\Theta(n)$ worst-case for deterministic evaluation algorithms, there is a simple *randomized* algorithm for evaluating NAND trees that, on expectation, does less than $\Theta(n)$ work. The idea is simple: use the same algorithm as above, but choose which subtree to evaluate first uniformly at random. We'll call this the *random-first algorithm*. More concretely:

- If the tree is a single leaf node, return the value of that node.
- Otherwise:
- Choose one of the subtrees of the root at random and evaluate it.
- If the value is 0, return 1.
- Otherwise, recursively evaluate the other subtree.
- If the value is 0, return 1; otherwise return 0.

To determine the runtime of the random-first algorithm, we will introduce *two* recurrence relations. Let $T_0(n)$ be the *expected* runtime of the random-first algorithm on a tree with n leaf nodes assuming the root evaluates to 0. Let $T_1(n)$ be the *expected* runtime of the random-first algorithm on a tree with n leaf nodes assuming the root evaluates to 1.

ii. Prove that the following recurrence relations for $T_0(n)$ and $T_1(n)$ are correct:

$$T_0(1) \le \Theta(1)$$
$$T_0(n) \le 2T_1(n / 2) + \Theta(1)$$

$$T_1(1) \le \Theta(1)$$
$$T_1(n) \le \tfrac{1}{2}T_1(n / 2) + T_0(n / 2) + \Theta(1)$$

iii. **(**It turns out that $T_1(n) \le T_0(n)$, though it's somewhat difficult to formally establish this. Using this fact, prove that $T_0(n) = O(n^\varepsilon)$ for some $\varepsilon < 1$. You can assume $n = 4^k$ for some natural number k. *(Hint: Write* $T_0(n)$ *in terms of itself.)*

Your result from (iii) proves that the random-first algorithm has expected sublinear runtime on *all* inputs, since $T_1(n) \le T_0(n) = O(n^\varepsilon) = o(n)$. This is one of a few known problems where the best-randomized algorithm is more efficient on expectation than the best deterministic algorithm in the worst case.

The last part of this problem explores this question: what happens if you try to evaluate a randomly chosen NAND tree? The result is surprising.

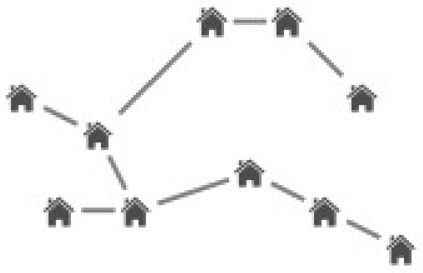
Let's say a *random NAND tree* with $n = 2^k$ leaves is a NAND tree where each leaf is independently assigned a value of 0 or 1 uniformly at random.

iv. Let $P_0(n)$ denote the probability that a random NAND tree with n leaves evaluates to 0 and $P_1(n)$ denote the probability that a random NAND tree with n leaves evaluates to 1. Write recurrence relations for $P_0(n)$ and $P_1(n)$ and briefly explain why your recurrences are correct.

The recurrence relations you came up with in (iv) can't be solved using the techniques we've developed in this course, but you can easily write a short computer program to determine their values by writing out $n = 2^k$ and evaluating the recurrence for increasing values of k. If you do, you'll find that when $k \ge 15$, $P_0(n)$ is extremely close to 1 if k is even and $P_1(n)$ is extremely close to 1 if k is odd. Consequently, the algorithm “return the height of the tree modulo 2” returns the right answer with high probability in time $\Theta(\log n)$, even though it never actually evaluates the tree!

Topic 3: Building Roads to Connect Cities

Problem Introduction



In this problem, the goal is to build roads between some pairs of the given cities such that there is a path between any two cities and the total length of the roads is minimized.

Problem Description

Task. Given *n* points on a plane, connect them with segments of minimum total length such that there is a path between any two points. Recall that the length of segments with endpoints $(x_1, y_1)$and (x_2, y_2) is equal to $\sqrt{(x_1, y_1)^2+(x_2, y_2)^2}$

Input format. The first line contains the number *n* of points. Each of the following *n* lines defines a point(x_i, y_i).

Constraints. $1 \leq n \leq 200$; $-10^3 \leq x_i, y_i \leq 10^3$ are integers. All points are pairwise different, no three points lie on the same line.

Output Format. Output the minimum total length of segments. The absolute value of the difference between the answer of your program and the optimal value should be at most 10^{-6}. To ensure this, output your answer with at least seven digits after the decimal point(otherwise your answer, while being computed correctly, can turn out to be wrong because of rounding issues).

Time Limits.

Language	**C**	**C++**	**Java**	**Python**	**C#**	**Haskell**	**JavaScript**	**Ruby**	**Scala**
Time(sec)	2	2	3	10	3	4	10	6	6

Sample 1.

Input:

4

0 0

0 1

1 0

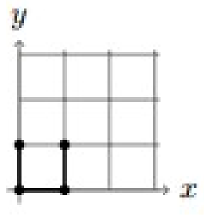
1 1

Memory limit. 512MB.

Output:

3.000000000

An optimal way to connect these four points is shown below. Note that there exists other ways of connecting these points by segments of total weight 3.



Sample 2.

Input:

5

0 0

0 2

1 1

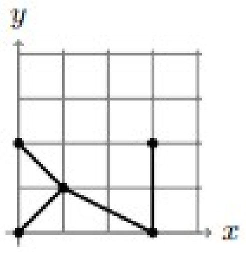
3 0

3 2

Output:

7.064495102

An optimal way to connect these five points is shown below.



The total length here is equal to $2\sqrt{2}+\sqrt{5}+2$.

Topic 4: Computing Prime Paths and Test-Paths

Problem Introduction

Control Flow Graphs (CFGs) are essential representations of program code that help analyze the different execution paths within a program. In this project, we aim to compute prime paths from a given CFG. Prime paths are fundamental in understanding program behavior, identifying critical paths, and optimizing code. We will explore topics such as side-trips, detours, and execution paths to achieve this goal.

A **control flow graph (CFG)** is a graphical representation of a program's basic blocks and their interconnections. Each basic block represents a sequence of non-compound statements that execute together. CFGs are widely used in code-checking tools, compilers, and software analysis.

Problem Description

I. **Constructing the CFG (Optional):**
   - Build a CFG from a given program using basic blocks.
   - Identify entry and exit points in the CFG.

II. **Prime Path Computation:**
   - Define prime paths as sequences of basic blocks that cover all possible execution paths. The exact definition is available at (Ammann & Offutt, 2016) .
   - Develop an algorithm to find prime paths efficiently.
   - Consider side trips (additional paths) and detours (revisiting blocks) during path computation.

III. **Path Coverage Analysis:**
   - Evaluate the coverage of prime paths in the CFG.
   - Investigate the impact of different paths on program behavior.

Methodology

i. **CFG Construction (Optional):**

Parse the program code to extract basic blocks.

Create a directed graph with basic blocks as nodes and edges representing control flow.

ii. **Prime Path Algorithm:**

Adapt existing algorithms (e.g., Floyd-Warshall, Tarjan) to find prime paths.

Handle loops, conditionals, and branching effectively.

IV. Coverage Analysis:

Implement path traversal to validate prime paths.

Measure coverage and identify uncovered paths.

iii. Computational complexity analysis

Discuss the time and space complexity of the proposed algorithms using asymptotic notations.

Sample outputs are available at *https://cs.gmu.edu:8443/offutt/coverage/GraphCoverage*

Topic 5: Automated Detection of Design Patterns and Smells in Class Diagrams

Problem Description

Class diagrams are fundamental in software modeling, representing the static structure of a system. In this project, we aim to develop an automated approach to identify common design patterns and detect code smells within class diagrams. By leveraging graph-based techniques, we will analyze the relationships between classes and uncover potential issues.

Class diagrams provide a visual representation of classes, their attributes, and associations. Detecting design patterns (such as Singleton, Factory Method, and Observer) and identifying code smells (such as God Classes, Feature Envy, and Inappropriate Intimacy) in class diagrams is crucial for maintaining software quality (Fowler & Beck, 2018) .

Objectives

1. **Graph Representation:**
   - Convert class diagrams into directed labeled graphs (DLGs).
   - Define nodes (classes) and edges (associations) in the DLG.
2. **Design Pattern Detection:**
   - Implement algorithms to recognize common design patterns.
   - Explore graph traversal techniques to identify pattern structures.
3. **Code Smell Identification:**
   - Define metrics for code smells (e.g., high coupling, low cohesion).
   - Analyze the DLG to detect potential code smell instances.
4. **Algorithmic Questions:**
   - Discuss the time and space complexity of the detection algorithms.

  - Investigate trade-offs between accuracy and efficiency.
  - Propose improvements or optimizations.

Methodology

1. **Graph Construction:**
   - Parse class diagrams (UML (XMI) or other formats) to create DLGs.
   - Represent classes as nodes and associations as edges.
2. **Design Pattern Detection Algorithms:**
   - Implement algorithms for detecting common patterns (e.g., Singleton, Factory Method). By representing design patterns as subgraphs, you can use subgraph isomorphism algorithms to identify occurrences of these patterns in class diagrams
   - Consider pattern variations and edge cases.
3. **Code Smell Metrics:**
   - Define metrics (e.g., coupling, cohesion) based on graph properties.
   - Evaluate class relationships to identify smells.
4. **Evaluation and Validation:**
   - Apply the approach to real-world class diagrams.
   - Validate results against manually labeled patterns and smells.

Topic 6: Constrained Scheduling

Suppose you have a supercomputer that can run jobs one at a time. You have a set of jobs J that you need to run and want to determine the best order in which to run them. Not all jobs take the same amount of time to complete; specifically, job j_k takes time t_k to complete. Each job must run to completion once started, so you can't pause or stop a job after starting it.

Certain jobs depend on results computed by other jobs, so you cannot run the jobs in a completely arbitrary order. Specifically, you have a DAG $G = (J, E)$ whose nodes are the jobs J and where each edge (j_i, j_k) indicates that job j_i must be run before job j_k.

Under these restrictions, it's easy to schedule all the jobs as efficiently as possible: just topologically sort the DAG and run the jobs in that order. Of course, there's a catch. Associated with each job j_k is a cost function $c_k(t)$ denoting the cost of completing job j_k at time t. These functions are monotonically increasing, so for any job j_k and any $\varepsilon > 0$, we have $c_k(t) < c_k(t + \varepsilon)$. Your task is to find a way of ordering all of the jobs on the supercomputer so that all constraints are satisfied and the total cost is as low as possible. Specifically, you want to minimize

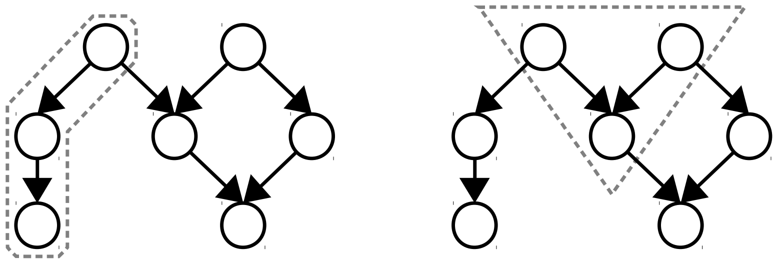
$$\sum_{j_k \in J} c_k(f(j_k))$$

Where $f(j_k)$ denotes the time at which job j_k finishes. This problem is known to be **NP**-hard.

A naïve algorithm for this problem is to try out every possible topological ordering of the DAG and find the ordering with the least total cost, but this algorithm can be incredibly slow.

i. Prove that for all $n \geq 0$, there is a DAG with n nodes and $\Omega(n!)$ topological orderings. This shows the naïve algorithm has worst-case runtime $\Omega(n!)$.

Fortunately, we can improve upon the naïve algorithm using dynamic programming. Let's call a set $S \subseteq J$ a *feasible set* iff for every $j_k \in S$, if there is a path from j_i to j_k in G (i.e. j_k depends on j_i), then $j_i \in S$. Intuitively, a feasible set is a set of jobs that can be scheduled without missing any prerequisites. For example, in the following DAGs, the indicated nodes are feasible sets:



For any feasible set *S*, let *LAST*(*S*) denote the set of all jobs $j_k \in S$ such that $(j_k, j_i) \notin E$ for any $j_i \in S$. In other words, *LAST*(*S*) consists of all jobs in *S* that no other jobs depend on. ii. **(3 Points)** Prove that if *S* is a feasible set, then $S - \{j\}$ is feasible for any $j \in LAST(S)$.

For any feasible set *S*, let OPT(*S*) denote the optimal cost of scheduling the jobs in set *S*.

iii. Prove that in any optimal schedule for the jobs in *S*, the supercomputer is never idle before all jobs have been completed (*i.e.* until all jobs have finished executing, the supercomputer is always executing some job.)

iv. Write a recurrence relation for OPT(*S*), then prove that your recurrence relation is correct.

Given a recurrence relation for OPT(*S*), it's possible to find the cost of an optimal schedule by using the following dynamic programming algorithm:

- Let DP be a table of size 2^n.
- For each subset $S \subseteq J$, in an appropriate order:
- If *S* is feasible, fill in DP[*S*] based on the recurrence from (iv).
- Return DP[*J*].

If we assume each function c_k can be evaluated in time O(1), then (with the right recurrence relation for OPT(*S*)) it's possible to fill each entry of DP in time $O(n + m)$. It's also possible to check whether a set is feasible in time $O(n + m)$. This means that the overall runtime for this algorithm is $O(2^n (n + m))$, which is significantly better than the $\Omega(n!)$ worst-case of the naïve algorithm!

References

Ammann, P., & Offutt, J. (2016). Introduction to software testing. Cambridge University Press. https://doi.org/DOI: 10.1017/9781316771273

Fowler, M., & Beck, K. (2018). *Refactoring: improving the design of existing code* (Second Edi). Addison-Wesley. https://refactoring.com/