Analysis and Design of Algorithms

Divide-and-Conquer: Searching in an Array

Instructor: **Morteza Zakeri**

Slide by: Neil Rhodes

Modified by: Morteza Zakeri

Outline

1. Main Idea of Divide-and-Conquer
2. Linear Search
3. Binary Search

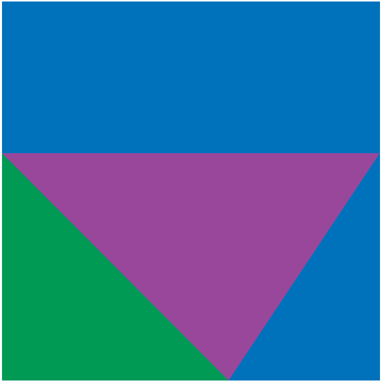
a problem to be solved

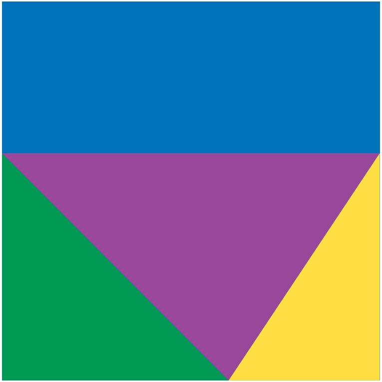
Divide: Break into non-overlapping subproblems of the same type

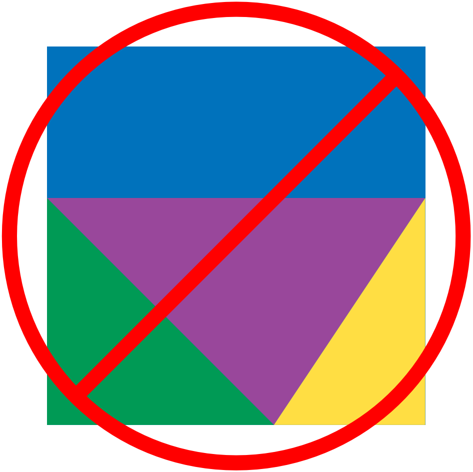
Divide: Break into non-overlapping subproblems of the same type

Divide: Break into non-overlapping subproblems of the same type

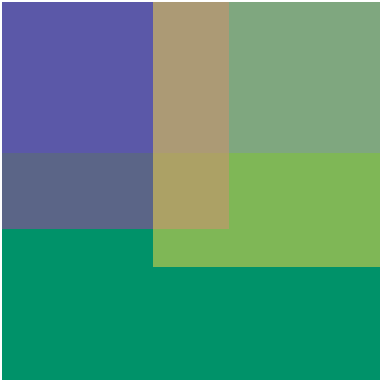
Divide: Break into non-overlapping subproblems of the same type

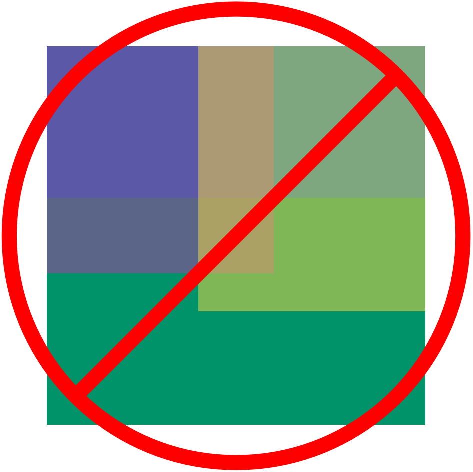






not the
same type





overlapping

Divide: break apart

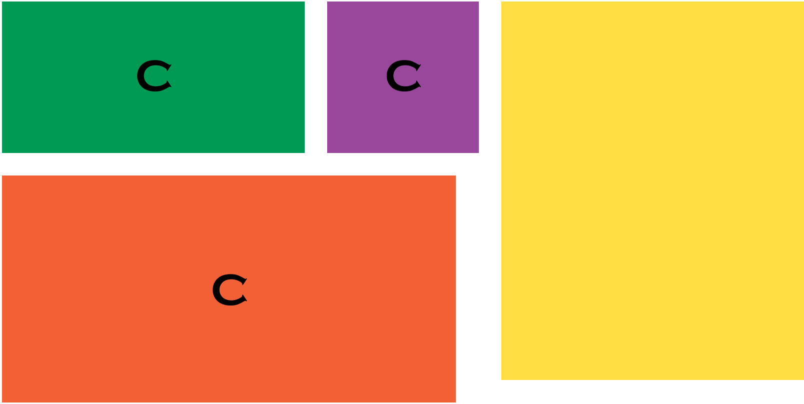
Divide: break apart

Conquer: solve subproblems

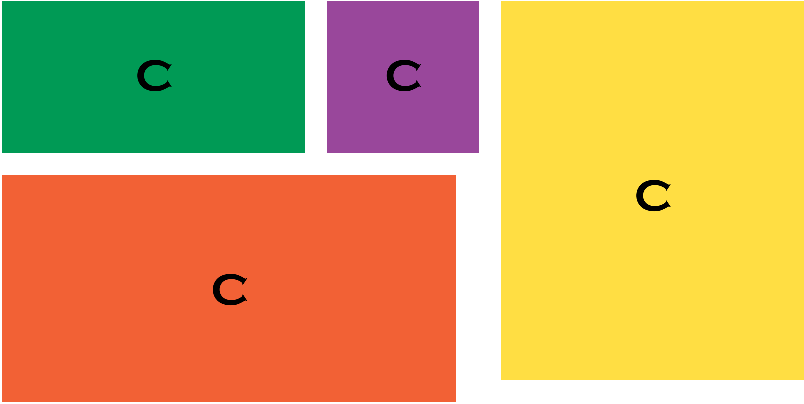
Conquer: solve subproblems

Conquer: solve subproblems

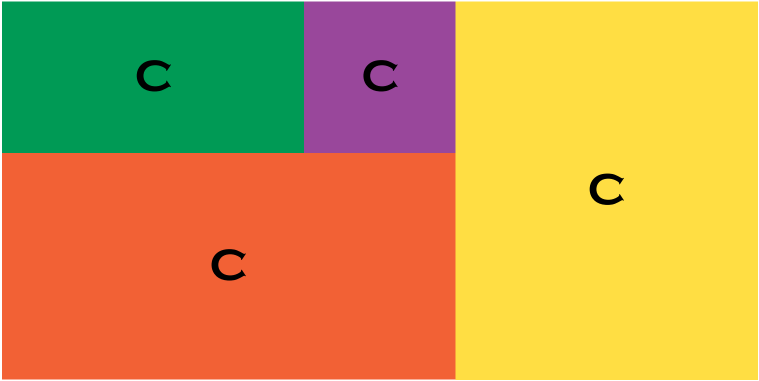
Conquer: solve subproblems



Conquer: solve subproblems



Conquer: combine



C

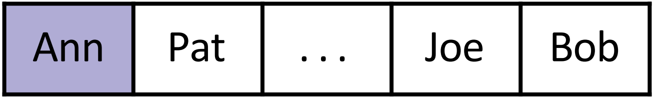
1. Break into non-overlapping subproblems of the same type
2. Solve subproblems
3. Combine results

Outline

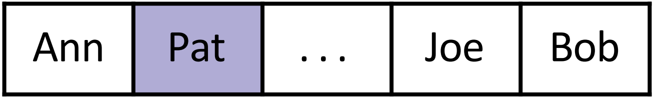
Linear Search in Array

Ann	Pat	. . .	Joe	Bob

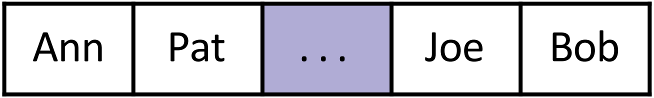
Linear Search in Array



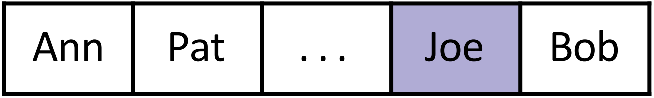
Linear Search in Array



Linear Search in Array



Linear Search in Array



Linear Search in Array

Ann	Pat	. . .	Joe	Bob

Linear Search in Array

Ann	Pat	. . .	Joe	Bob

Real-life Example

english	french	italian	german	spanish
house	maison	casa	Haus	casa
car	voiture	auto	Auto	auto
table	table	tavola	Tabelle	mesa

Searching in an array

Input: An array A with n elements.
A key k.

Output: An index, i, where $A[i] = k$.
If there is no such i, then
NOT_FOUND.

Recursive Solution

```
LinearSearch(A, low, high, key)
```

Recursive Solution

```
LinearSearch(A, low, high, key)
if high < low:
  return NOT_FOUND
if A[low] = key:
  return low
```

Recursive Solution

```
LinearSearch(A, low, high, key)
if high < low:
  return NOT_FOUND
if A[low] = key:
  return low
return LinearSearch(A, low + 1, high, key)
```

Recursive Solution

```
LinearSearch(A, low, high, key)

if high < low:
  return NOT_FOUND
if A[low] = key:
  return low
return LinearSearch(A, low + 1, high, key)
```

Definition

A recurrence relation is an equation recursively defining a sequence of values.

Definition

A recurrence relation is an equation recursively defining a sequence of values.

Fibonacci recurrence relation

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{if } n > 1 \end{cases}$$

Definition

A recurrence relation is an equation recursively defining a sequence of values.

Fibonacci recurrence relation

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{if } n > 1 \end{cases}$$

$$0, 1, 1, 2, 3, 5, 8, \ldots$$

LinearSearch(*A*, *low*, *high*, *key*)

```
if high < low:
  return NOT_FOUND
if A[low] = key:
  return low
return LinearSearch(A, low + 1, high, key)
```

LinearSearch(A, low, $high$, key)

```
if high < low:
  return NOT_FOUND
if A[low] = key:
  return low
return LinearSearch(A, low + 1, high, key)
```

Recurrence defining worst-case time:

$$T(n) = T(n - 1) + c$$

LinearSearch(A, low, $high$, key)

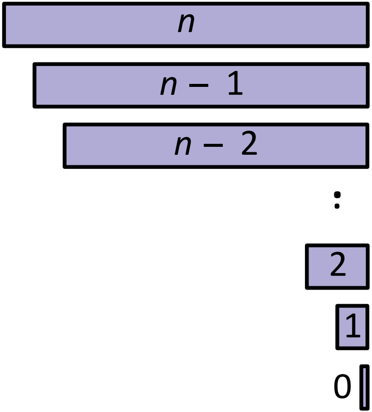
```
if high < low:
    return NOT_FOUND
if A[low] = key:
    return low
return LinearSearch(A, low + 1, high, key)
```

Recurrence defining worst-case time:

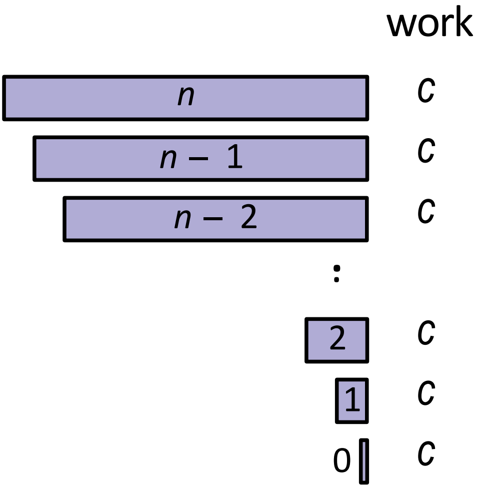
$$T(n) = T(n-1) + c$$

$$T(0) = c$$

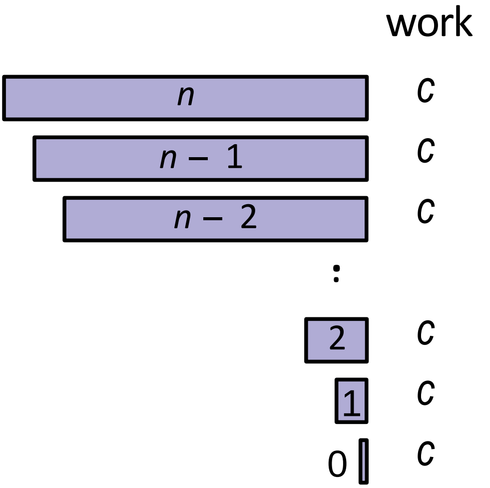
Runtime of Linear Search



Runtime of Linear Search



Runtime of Linear Search



Total: $\sum_{i=0}^{n} c = \Theta(n)$

Iterative Version

```
LinearSearchIt(A, low, high, key)
for i from low to high:
    if A[i] = key:
        return i
return NOT_FOUND
```

Summary

- Create a recursive solution

Summary

- Create a recursive solution
- Define a corresponding recurrence relation, T

Summary

- Create a recursive solution
- Define a corresponding recurrence relation, T
- Determine $T(n)$: worst-case runtime

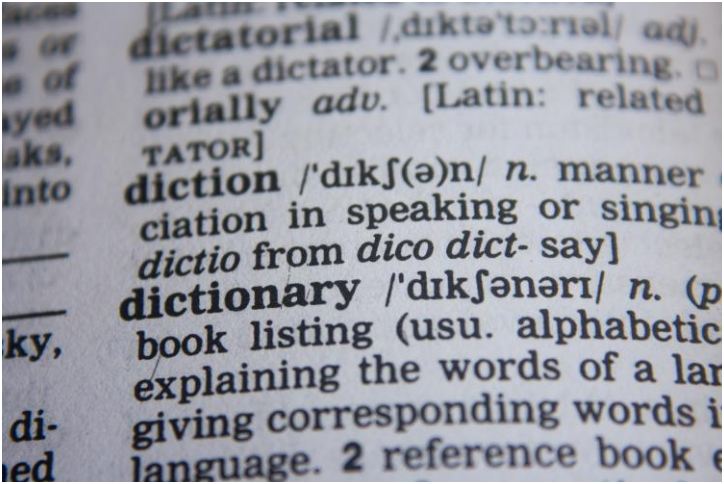
Summary

- Create a recursive solution
- Define a corresponding recurrence relation, T
- Determine $T(n)$: worst-case runtime
- Optionally, create iterative solution

Outline

1. Main Idea of Divide-and-Conquer
2. Linear Search
3. Binary Search

Searching Sorted Data



Searching in a sorted array

Input: A sorted array $A[low \ldots high]$

$(\forall low \leq i < high: A[i] \leq A[i+1])$.

A key k.

Output: An index, i, $(low \leq i \leq high)$ where $A[i] = k$.

Otherwise, the greatest index i, where $A[i] < k$.

Otherwise $(k < A[low])$, the result is $low - 1$.

Searching in a Sorted Array

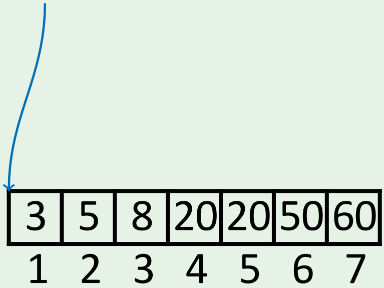
Example

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

search(2) → 0

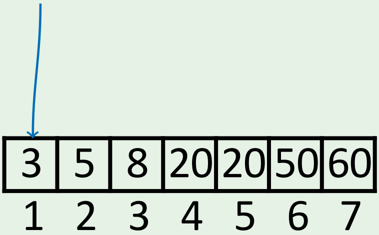


Searching in a Sorted Array

Example

search(2) → 0

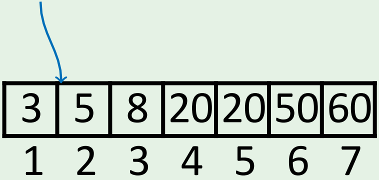
search(3) → 1



Searching in a Sorted Array

Example

search(2) → 0
search(3) → 1
search(4) → 1



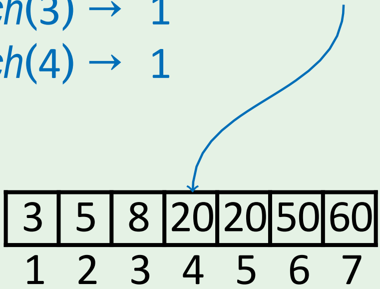
Searching in a Sorted Array

Example

search(2) → 0 *search*(20) → 4

search(3) → 1

search(4) → 1



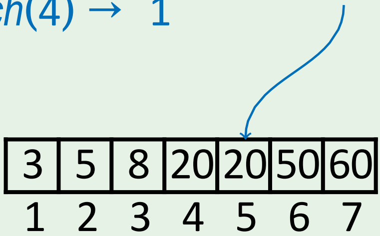
Searching in a Sorted Array

Example

search(2) → 0 *search*(20) → 4

search(3) → 1 *search*(20) → 5

search(4) → 1



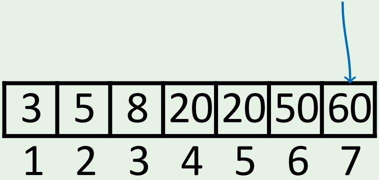
Searching in a Sorted Array

Example

search(2) → 0   *search*(20) → 4

search(3) → 1   *search*(20) → 5

search(4) → 1   *search*(60) → 7



Searching in a Sorted Array

Example

$search(2) \to 0$ $search(20) \to 4$

$search(3) \to 1$ $search(20) \to 5$

$search(4) \to 1$ $search(60) \to 7$

$search(70) \to 7$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

BinarySearch(*A*, *low*, *high*, *key*)

BinarySearch(*A*, *low*, *high*, *key*)

```
if high < low:
  return low − 1
```

BinarySearch(*A*, *low*, *high*, *key*)

```
if high < low:
    return low − 1
mid ← ⌊low + (high−low)/2⌋
```

$$mid \leftarrow \left\lfloor low + \frac{high-low}{2} \right\rfloor$$

BinarySearch(A, low, $high$, key)

if $high < low$:

  return $low - 1$

$mid \leftarrow \left\lfloor low + \frac{high - low}{2} \right\rfloor$

if $key = A[mid]$:

  return mid

BinarySearch(A, low, $high$, key)

```
if high < low:
    return low − 1
mid ← ⌊low + (high−low)/2⌋
if key = A[mid]:
    return mid
else if key < A[mid]:
    return BinarySearch(A, low, mid − 1, key)
```

BinarySearch$(A, low, high, key)$

```
if high < low:
    return low − 1
mid ← ⌊low + (high−low)/2⌋
if key = A[mid]:
    return mid
else if key < A[mid]:
    return BinarySearch(A, low, mid − 1, key)
else:
    return BinarySearch(A, mid + 1, high, key)
```

Example: Searching for the key 50

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

Example: Searching for the key 50

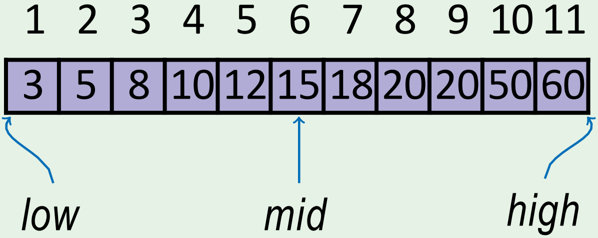
BinarySearch(A, 1, 11, 50)

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

low *high*

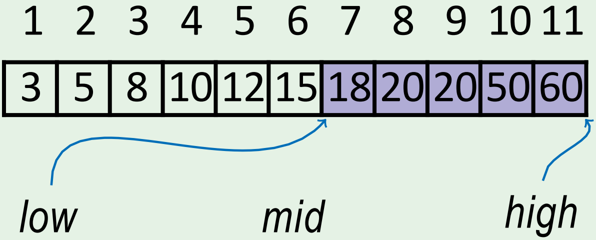
Example: Searching for the key 50

BinarySearch(A, 1, 11, 50)



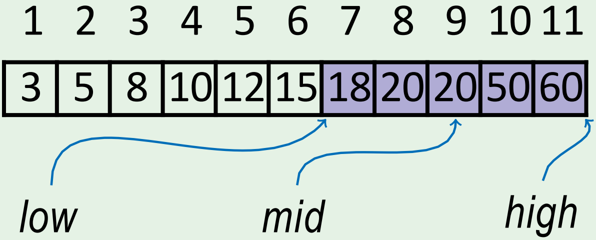
Example: Searching for the key 50

BinarySearch(A, 1, 11, 50)
BinarySearch(A, 7, 11, 50)



Example: Searching for the key 50

BinarySearch(A, 1, 11, 50)
BinarySearch(A, 7, 11, 50)



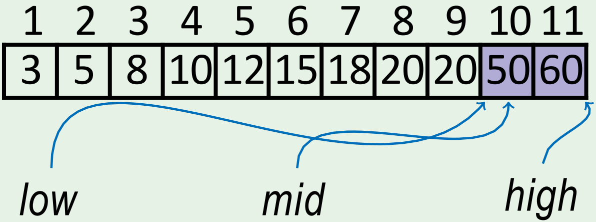
Example: Searching for the key 50

BinarySearch(A, 1, 11, 50)
BinarySearch(A, 7, 11, 50)
BinarySearch(A, 10, 11, 50)

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

Example: Searching for the key 50

BinarySearch(A, 1, 11, 50)
BinarySearch(A, 7, 11, 50)
BinarySearch(A, 10, 11, 50)



Example: Searching for the key 50

$\mathrm{BinarySearch}(A, 1, 11, 50)$
$\mathrm{BinarySearch}(A, 7, 11, 50)$
$\mathrm{BinarySearch}(A, 10, 11, 50) \rightarrow 10$

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

Summary

- Break problem into non-overlapping subproblems of the same type.

Summary

- Break problem into non-overlapping subproblems of the same type.
- Recursively solve those subproblems.

Summary

- Break problem into non-overlapping subproblems of the same type.
- Recursively solve those subproblems.
- Combine results of subproblems.

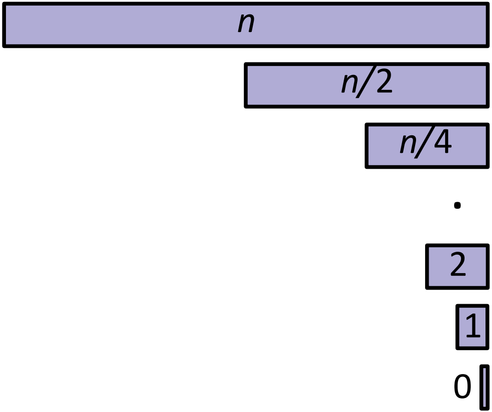
BinarySearch$(A, low, high, key)$

```
if high < low:
    return low − 1
mid ← ⌊low + (high − low)/2⌋
if key = A[mid]:
    return mid
else if key < A[mid]:
    return BinarySearch(A, low, mid − 1, key)
else:
    return BinarySearch(A, mid + 1, high, key)
```

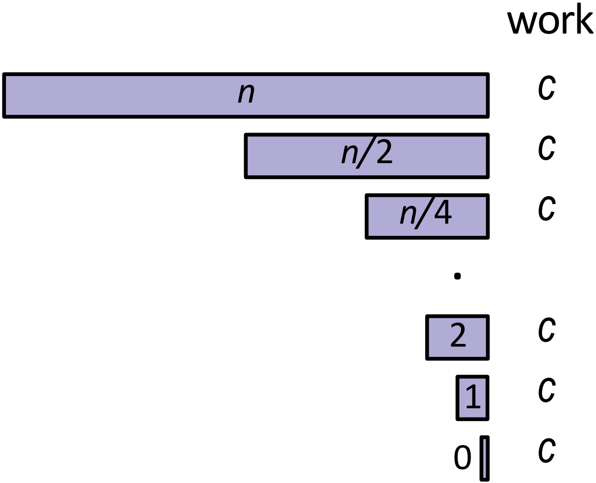
Binary Search Recurrence Relation

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + c$$
$$T(0) = c$$

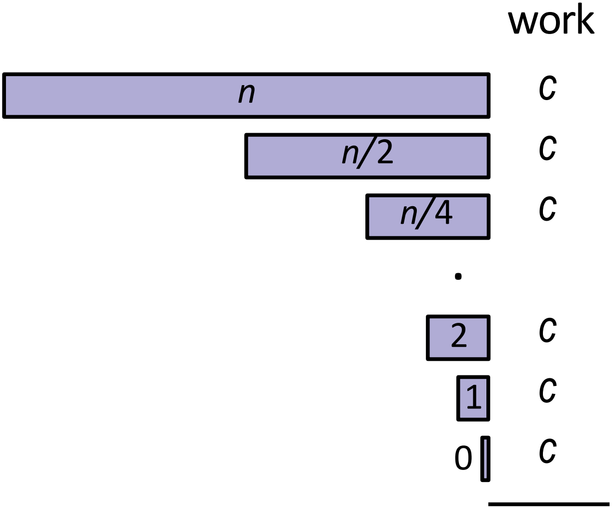
Runtime of Binary Search



Runtime of Binary Search



Runtime of Binary Search



Total: $\sum_{i=0}^{\log_2 n} c = \Theta(\log_2 n)$

Iterative Version

BinarySearchIt(A, low, $high$, key)

while $low \leq high$:

  $mid \leftarrow \lfloor low + \frac{high - low}{2} \rfloor$

Iterative Version

BinarySearchIt($A, low, high, key$)

```
while low ≤ high:
    mid ← ⌊low + (high−low)/2⌋
    if key = A[mid]:
        return mid
```

Iterative Version

```
BinarySearchIt(A, low, high, key)

while low ≤ high:
    mid ← ⌊low + (high−low)/2⌋
    if key = A[mid]:
        return mid
    else if key < A[mid]:
        high = mid − 1
```

Iterative Version

```
BinarySearchIt(A, low, high, key)

while low ≤ high:
   mid ← ⌊low + (high−low)/2⌋
   if key = A[mid]:
      return mid
   else if key < A[mid]:
      high = mid − 1
   else:
      low = mid + 1
```

Iterative Version

BinarySearchIt(A, low, $high$, key)

```
while low ≤ high:
    mid ← ⌊low + (high−low)/2⌋
    if key = A[mid]:
        return mid
    else if key < A[mid]:
        high = mid − 1
    else:
        low = mid + 1
return low − 1
```

Real-life Example

english	french	italian	german	spanish
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

Real-life Example

english (sorted)	**french** (sorted)	**italian** (sorted)	**german** (sorted)	**spanish** (sorted)
chair	chaise	casa	Haus	casa
house	bouton	foruncolo	Pickel	espenilla
pimple	maison	sedia	Sessel	silla

Real-life Example

english	french	italian	german	spanish
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

english sorted

2
1
3

spanish sorted

1
3
2

Real-life Example

english	**french**	**italian**	**german**	**spanish**
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

english sorted
2
1
3

spanish sorted
1
3
2

Real-life Example

english	french	italian	german	spanish
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

english sorted
2
1
3

spanish sorted
1
3
2

Real-life Example

english	french	italian	german	spanish
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

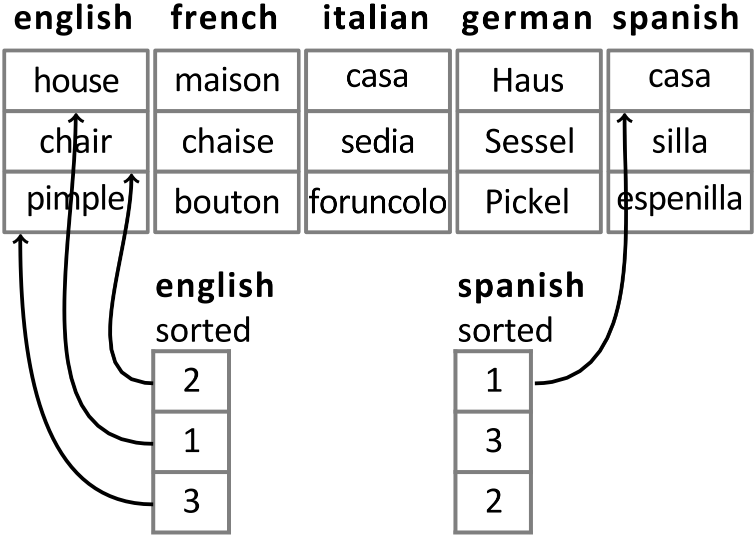
english sorted

2
1
3

spanish sorted

1
3
2

Real-life Example



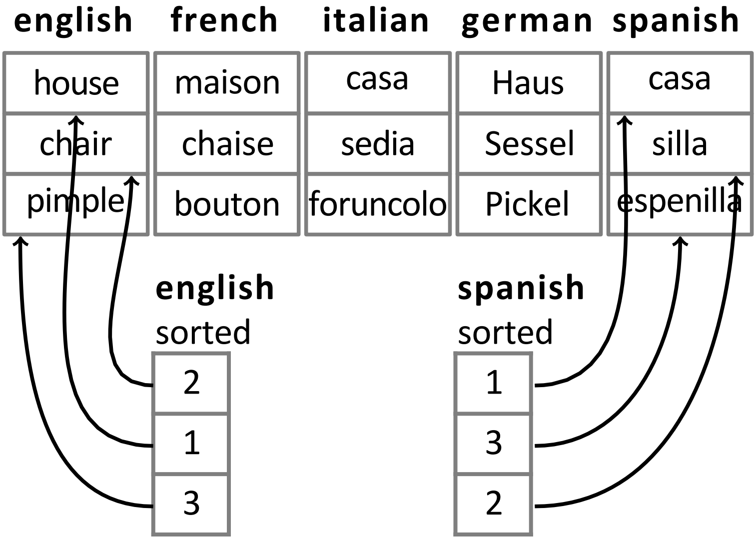
Real-life Example

english	french	italian	german	spanish
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

english sorted
2
1
3

spanish sorted
1
3
2

Real-life Example



Summary

The runtime of binary search is $\Theta(\log n)$.